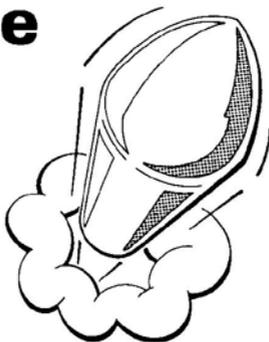


Projectile Motion



2

To start the unit, proceed as follows:

If your machine has a single drive, load disc 1, hold down the shift key and press the break key.

If your machine has a double drive, load disc 1 into the upper drive and disc 2 into the lower. Then hold down the shift key and press the break key.

At various points, when you press the space bar, a card number will appear at the bottom of the screen. When this occurs, read the card indicated and then press the space bar again. The micro will wait until you make the second press in order to give you time to read the card.

At certain points, a loop is incorporated so that you can repeat a sequence, but with different data. In such cases, you will find that you may get the message to read a particular card more than once — every time you go round the loop, in fact. When this happens, read the card the first time but after that ignore the instruction — unless, of course, you want to read it again.

Sometimes a section of a unit may be complete in itself, and so does not require any additional information in card form. In such a section, no card numbers will appear.

The basic equations for projectiles with no resistance are the equations of motion for a particle moving with constant acceleration, and we assume that you are familiar with these. They are

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

When using these equations for projectiles, we consider the horizontal and vertical motions separately. Horizontally there is no force acting and so the horizontal component of the velocity remains constant and the horizontal distance travelled is simply this velocity component multiplied by the time that motion is taking place. In the picture you have just seen, this velocity was 60ms^{-1} and so, after t secs, $x = 60t$, x being the horizontal distance travelled. For the same situation, ie the initial velocity completely horizontal, what will be the expressions for \dot{y} and y at time t , y being the vertical distance travelled upwards? When you have decided, turn to card 2 to check your answer.

Projectile Motion	1	Card 1	
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$$\dot{y} = -gt$$

$$y = -\frac{1}{2}gt^2$$

Here the acceleration is g downwards and there is no initial vertical component of velocity. In the current situation $y = 300$ when the projectile lands. Taking $g = 9.81\text{ms}^{-2}$ (this value will be used throughout this unit) find the time taken for the projectile to land and the horizontal distance travelled. Your second result should agree with the value of R shown on the screen. Then PSB.

**Projectile
Motion**

1

Card 2

psb

Calculate the time taken for the projectile to fall freely for the height you have chosen. PSB.

Projectile Motion	1	Card 3	psb
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You will now be invited to choose a horizontal velocity in the range 10 to 30ms⁻¹. What will be the horizontal distance travelled for the value you intend to choose. PSB and so verify your answer.

Projectile Motion	1	Card 4	psb
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You should have noticed that the time taken for the projectile to land depends only on the vertical distance it has to fall, not on the horizontal velocity of projection. The micro will now ask you if you wish to consider energy in relation to the type of motion we have been studying. If your answer is 'No' do not read any more of this card, just type N. If your answer is 'Yes', finish reading this card before you actually type Y.

There are two forms of energy relevant to our work on projectiles — kinetic and potential. For a particle of mass m moving with speed v , the former is given by $\frac{1}{2}mv^2$. If the particle is at a height h above a datum level, the latter is given by mgh . When you type Y, the micro will repeat the last motion you have just seen and state the values of the two forms of energy at successive time intervals. Notice that the sum of the two forms remains constant throughout the motion.

As has been indicated on the screen, we now go on to consider what happens when a particle is projected from ground level at an angle to the horizontal. If the initial speed is u at an angle α to the horizontal, then the initial horizontal and vertical velocity components are $u \cos \alpha$ and $u \sin \alpha$. What will be the expressions for \dot{x} , \dot{y} , x and y at any subsequent time t ? Turn to card 7 to check your answers.

$$\dot{x} = u \cos \alpha$$

$$\dot{y} = u \sin \alpha - gt$$

$$x = ut \cos \alpha$$

$$y = ut \sin \alpha - \frac{1}{2}gt^2$$

PSB and make a note of the initial angle you choose and one of your initial velocities.

Use the equations on card 7 to find general results for the time of flight, range, maximum height reached and the time to reach this greatest height. Then calculate these items for the values you have noted. Check your general results from card 9 and your particular values on the micro.

Projectile Motion	2.1	Card 8	
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$$\text{Time of flight} = \frac{2u \sin \alpha}{g}$$

$$\text{Range} = \frac{2u^2 \sin \alpha \cos \alpha}{g} = \frac{u^2 \sin 2\alpha}{g}$$

$$\text{Time to maximum height} = \frac{u \sin \alpha}{g}$$

$$\begin{aligned} \text{Maximum height} &= \frac{u^2 \sin^2 \alpha}{g} - \frac{1}{2}g \frac{u^2 \sin^2 \alpha}{g^2} \\ &= \frac{u^2 \sin^2 \alpha}{2g} \end{aligned}$$

You will notice that all these values depend on u and α . What, for a given value of u , will be the value of α that gives maximum range and what will be this maximum range? Turn to card 10 to check.

$\frac{u^2 \sin 2\alpha}{g}$ is a maximum when $\sin 2\alpha = 1$, ie when $\alpha = \frac{\pi}{4}$.

Then maximum range = u^2/g .

PSB.

Now let us see how the results just illustrated can be worked out theoretically. The first thing that we require is the equation of the path of the projectile. Working with a general initial speed of u (rather than 100), the equation of the path can be found by eliminating t between the equations

$$x = ut \cos \alpha$$

$$\text{and } y = ut \sin \alpha - \frac{1}{2}gt^2$$

Carry out this elimination and see what you get. Then turn to card 12 to check your answer.

$y = x \tan \alpha - \frac{gx^2}{2u^2} \sec^2 \alpha$, and this is the equation of a parabola. By writing $\sec^2 \alpha = 1 + \tan^2 \alpha$, we have a quadratic in $\tan \alpha$. Solve this quadratic for $\tan \alpha$ and then turn to card 13.

$$\tan \alpha = \{u^2 \pm \sqrt{u^4 - g(gx^2 + 2u^2y)}\}/gx$$

There will thus be:

Two values of α if $u^4 - g(gx^2 + 2u^2y) > 0$

One value of α if $u^4 - g(gx^2 + 2u^2y) = 0$

No value of α if $u^4 - g(gx^2 + 2u^2y) < 0$

When there is only one value of α , the point (x, y) lies on the parabola of safety, which thus has the equation

$$u^4 - g(gx^2 + 2u^2y) = 0$$

We can now tie in these results with those obtained in card 10.

Taking the case where there is just one value of α and putting $y = 0$ in the equation of the parabola of safety gives

$$u^4 - g^2x^2 = 0$$

$$\text{ie } x = u^2/g$$

which is the maximum range. Then $\tan \alpha = 1$ and so $\alpha = \frac{\pi}{4}$.

Next, looking at the case where there are two values of α , say α_1 and α_2 you can easily verify that, if $y = 0$, $\tan \alpha_1 = \frac{1}{\tan \alpha_2}$.

Therefore $\alpha_1 = \frac{\pi}{2} - \alpha_2$ or $\alpha_1 - \frac{\pi}{4} = \frac{\pi}{4} - \alpha_2$. Thus, if we wish to hit a given

point within the parabola of safety for which $y = 0$, there are two angles of projection, equally inclined to 45° .

Now PSB and answer 'Yes' when asked if you want random targets. Then, for each, use your knowledge to predict whether or not you can hit the target and to calculate the angles of projection.

Using the formula for $\tan \alpha$ on card 13, verify the angles quoted on the screen. Note that these are only integer values, so if the calculated values give the range 11.07 to 18.79, the screen will show 12 to 18.

Projectile Motion	3.2	Card 14	psb
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Answer Y when you are asked if you want another example and, when it appears, calculate the range of values of α which will give you success. Then check your values using the micro.

Projectile Motion	3.2	Card 15	psb
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When you PSB, you will be asked if you wish to play cricket or tennis.
Have a game of tennis first and then cricket later on.

**Projectile
Motion**

3.3

Card 16

psb

When you PSB, you will be asked if you want to input fresh data. PSB, answer Y and PSB again. Then continue reading this card in conjunction with the diagram that will appear.

In a situation like this take the origin at the point from which the ball is hit, ie measure y from a distance H above ground level. Now assume that we decide on specific values for B and d , say $B = 24$, $d = 13$. Then effectively we have to choose u and α so that two conditions are satisfied. Can you say what these two conditions are? When you have decided, turn to the next card to check.

The conditions are:

- (i) When $x = 13$, $y > h - H$
- (ii) When $x = 24$, $y < -H$ or when $y = -H$, $x < 24$.

If you think for a moment you will realise that the alternatives in (ii) are really equivalent.

Now here we have H fixed at 1.5 and h at 1.0, hence the two conditions become:

- (i) When $x = 13$, $y > -0.5$.
- (ii) When $x = 24$, $y < -1.5$.

There are various ways in which we can now proceed:

- (a) Fix α and find a suitable u .
- (b) Fix u and find a suitable α .
- (c) Vary both u and α and find a suitable combination. In order to give some concrete working, suppose we decide to fix α at 18° . Have a go at working out u so as to clear the net. Turn to card 19 to check your working.

Using $y = x \tan \alpha - \frac{gx^2}{2u^2} \sec^2 \alpha$, with $x = 13$ and $\alpha = 18^\circ$,

$$\begin{aligned}y &= 13 \tan 18^\circ - \frac{9.81 \times 13^2}{2u^2} \sec^2 18^\circ \\ &= 4.224 - \frac{916.46}{u^2}\end{aligned}$$

But we need $y > -0.5$

$$\text{ie } 4.224 - \frac{916.46}{u^2} > -0.5$$

$$\text{ie } \frac{916.46}{u^2} < 4.724$$

$$\text{ie } u^2 > \frac{916.46}{4.724}$$

$$\text{ie } u > 13.93$$

Also, when $x = 24$, $y < -1.5$. Work out u so that this condition is satisfied. (You will probably find this easier than working out u such that when $y = -1.5$, $x < 24$.) Then turn to card 20 to check.

We require

$$24 \tan 18^\circ - \frac{9.81 \times 24^2}{2u^2} \sec^2 18^\circ < -1.5$$

$$\text{ie } \frac{3123.6}{u^2} > 9.298$$

$$\text{ie } u^2 < \frac{3123.6}{9.298}$$

$$\text{ie } u < 18.34$$

Hence, for both conditions to be satisfied, we require

$$13.93 < u < 18.34$$

Now check these values on the micro, and then play some more tennis if you wish. Then have a go at cricket, first being a batsman.

We are not going to go all through the numerical working in this case as it is very similar to that for tennis. But what will be the conditions here corresponding to those in card 18 for tennis? Turn to card 22 to check.

(i) When $x = d$, $y > h - H$

(ii) When $x = B$, $y > -H$ or $y = -H$ when $x > B$.

When you have finished batting, try your hand at bowling.

What conditions will have to be satisfied here for the wicket keeper to catch the ball? Assume that he can place his hands at any height up to h . Turn to the next card to check.

When $x = d$, $-H < y < h - H$.

**Projectile
Motion**

3.3

Card 24

psb

Calculate the time of flight and range for the angle of projection you are going to choose next. Verify that you agree with the micro. PSB.

Projectile Motion	4.1	Card 25	psb
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By using the formula in card 13 for $\tan \alpha$, check the results quoted on the screen. Then PSB.

Projectile Motion	4.2	Card 26	psb
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You will notice that the x and y axes are shown on the screen as being along and perpendicular to the slope. When dealing with inclined planes we work with reference to these directions instead of horizontally and vertically.

What will be the components of acceleration in these two directions? When you have decided, turn to card 28 to check.

Projectile Motion	5.1	Card 27	
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$-g \sin \beta \nearrow$ and $-g \cos \beta \nwarrow$.

What will these values give for \dot{x} , \dot{y} , x and y in terms of t for a speed of projection u at an angle α to the plane? Turn to card 29 to check your results.

$$\dot{x} = u \cos \alpha - gt \sin \beta$$

$$\dot{y} = u \sin \alpha - gt \cos \beta$$

$$x = ut \cos \alpha - \frac{1}{2}gt^2 \sin \beta$$

$$y = ut \sin \alpha - \frac{1}{2}gt^2 \cos \beta$$

How will you use these equations to calculate the three quantities shown at the bottom of the screen? Turn to card 30 to check your answers.

**Projectile
Motion**

5.1

Card 29

The time of flight is obtained by putting $y = 0$. Then

$$t = \frac{2u \sin \alpha}{g \cos \beta}$$

The formula for x then gives

$$\begin{aligned} x &= \frac{2u^2 \sin \alpha \cos \alpha}{g \cos \beta} - \frac{2u^2 \sin^2 \alpha \sin \beta}{g \cos^2 \beta} \\ &= \frac{2u^2 \sin \alpha \cos (\alpha + \beta)}{g \cos^2 \beta} \end{aligned}$$

The maximum value of y occurs when $\dot{y} = 0$. Then $t = \frac{u \sin \alpha}{g \cos \beta}$

$$\begin{aligned} \text{and } y &= \frac{u^2 \sin^2 \alpha}{g \cos \beta} - \frac{u^2 \sin^2 \alpha}{2g \cos \beta} \\ &= \frac{u^2 \sin^2 \alpha}{2g \cos \beta} \end{aligned}$$

Use these equations to check that the micro has got its arithmetic correct. Then PSB.

What differences will occur in the results obtained in cards 28-30 (for motion *up* an inclined plane) for the case that we now have? When you have worked these out, turn to card 32 to check.

**Projectile
Motion**

5.2

Card 31

$$\ddot{x} = g \sin \beta$$

$$\dot{x} = u \cos \alpha + gt \sin \beta$$

$$x = ut \cos \alpha + \frac{1}{2}gt^2 \sin \beta$$

\ddot{y} , \dot{y} and y remain unchanged.

Time of flight and maximum y remain unchanged.

Value of x when $y = 0$ is $\frac{2u^2 \sin \alpha \cos (\alpha - \beta)}{g \cos^2 \beta}$

You will notice that all of these results can be obtained by replacing β by $-\beta$ in the corresponding results for motion up the plane. When you PSB you will be invited to select a set of values. When you have done so, calculate the time of flight, and maximum x and y for the values you have chosen. PSB.

Can you decide what angle will give a maximum range for the data you have entered? Try to obtain this analytically now, and then check your result by PSB. If you have forgotten what data you entered, press ESCAPE and start again, making a note of it as you go along.

**Projectile
Motion**

5.3

Card 33

psb

Do you agree with the result? If so, PSB. If not, go through the following:

Projection up plane:

x is a maximum when $\frac{2u^2 \sin \alpha \cos (\alpha + \beta)}{g \cos^2 \beta}$ is a maximum

For given u and β , this is a maximum when

$$\sin \alpha \cos (\alpha + \beta)$$

is a maximum. What value of α will ensure this? Find it and then check your result by turning to card 35.

Differentiating with respect to α , we require

$$\cos \alpha \cos (\alpha + \beta) - \sin \alpha \sin (\alpha + \beta) = 0$$

ie $\cos \alpha \cos (\alpha + \beta) = \sin \alpha \sin (\alpha + \beta)$

ie $\cot \alpha = \tan (\alpha + \beta)$

ie $\tan (\alpha + \beta) = \tan \left[\frac{\pi}{2} - \alpha \right]$

ie $\alpha + \beta = \frac{\pi}{2} - \alpha$

ie $\alpha = \frac{\pi}{4} - \frac{\beta}{2}$

Similar working for projection down the plane gives the result

$$\alpha = \frac{\pi}{4} + \frac{\beta}{2}$$

PSB

If you have already studied impact, you should be able to find u_2 and α_2 given u_1 , α_1 and e . If you have, find them and check your results by turning to card 37. If you haven't turn to card 37 and read through what is given there. You should find it easy to follow.

Projectile Motion	6.1	Card 36	
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The projectile first hits the ground with velocity components $u_1 \cos \alpha_1 \rightarrow$ and $u_1 \sin \alpha_1 \downarrow$.

During impact, no force acts \rightarrow on the projectile and so its speed \rightarrow , ie $u_1 \cos \alpha_1$, remains unaltered.

By Newton's experimental law, its speed \downarrow , ie $u_1 \sin \alpha_1$, is changed in direction and multiplied by e . It thus becomes $eu_1 \sin \alpha_1 \uparrow$.

$$\begin{aligned}\text{Hence } u_2 &= \sqrt{(u_1 \cos \alpha_1)^2 + (eu_1 \sin \alpha_1)^2} \\ &= u_1 \sqrt{\cos^2 \alpha_1 + e^2 \sin^2 \alpha_1}\end{aligned}$$

$$\begin{aligned}\text{and } \tan \alpha_2 &= \frac{eu_1 \sin \alpha_1}{u_1 \cos \alpha_1} \\ &= e \tan \alpha_1\end{aligned}$$

Now PSB and calculate u_2 , α_2 , and u_3 , α_3 for the values you select to enter on the keyboard.

There is no basic difference between what is happening now and the previous case. Now, on impact, the velocity component \uparrow or \downarrow will be unchanged, while the component \rightarrow will be reversed and multiplied by e . For the values you have chosen, calculate where the projectile will hit the ground. Check your result against the micro. If you are in difficulties, a typical calculation is given in card 39.

Taking $u = 100 \text{ ms}^{-1}$, $\alpha = 45^\circ$, $e = 0.5$;

$$\text{Time to wall} = \frac{520}{100/\sqrt{2}} = 7.35 \text{ s}$$

Then $\text{vel} \rightarrow = 70.71$, $\text{vel} \uparrow = -1.43$

$$\text{and } y = \frac{100}{\sqrt{2}}t - \frac{1}{2}gt^2 = 254.74$$

After rebound, $\text{vel} \leftarrow = 35.36$, $\text{vel} \uparrow = -1.43$.

Additional time required is given by

$$-254.74 = -1.43t - \frac{1}{2} \times 9.81t^2$$

$$\text{ie } 4.905t^2 + 1.43t - 254.54 = 0$$

giving $t = 7.062 \text{ s}$

Distance travelled $\leftarrow = 249.72 \text{ m}$

$$\therefore x = 520 - 249.72 = 270.28 \text{ m}$$

Once again there is no basic difference between this case and the previous ones. On each impact, the velocity parallel to the plane is unaltered and that perpendicular to the plane is reversed and multiplied by e . (There are no cards for you to read when you come to Part 7 – resisted motion. In this part we are simply illustrating what happens without dealing with the mathematical theory.)

